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Toward a unified algebraic understanding of concepts of particle and collective motions in fermion many-body systems*

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Abstract

Toward a unified algebraic understanding of physical concepts of the so-called particle and collective motions, a new theory for unified description of both the motions is proposed with the use of time-dependent Hartree–Fock theory on a circle S^1. The theory simply and clearly elucidates a collective motion induced by a TD mean-field potential. It also describes symmetry breaking of fermion systems and successive occurrence of the collective motion due to recovery of the symmetry. The theoretical frame asserts that the Fock space of finite-dimensional fermions has an algebraic structure to be embedded into that of *infinite-dimensional fermions*.

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1. Introduction

The standard description of fermion many-body systems starts with the most basic approximation that is based on an independent-particle picture, i.e. self-consistent field (SCF) for motion of the fermions. The Hartree–Fock (HF) theory is typical of such approximations for ground states of the fermion systems. Excited states are treated with the well known random phase approximation (RPA). The time-dependent HF (TDHF) equation is a nonlinear equation due to its SCF character and has no unique solutions [1]. To go beyond a perturbative method with respect to collective variables to extract large-amplitude collective motions out of a fully parametrized TDHF manifold [2], we have proposed a unified aspect of the SCF

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viewpoint on group $U(n)$ and a τ-functional one on the group in soliton theory [3] (referred to as I). In this context we have aimed at having close connection between concepts of mean-field potential and the gauge of fermions inherent in the SCF method and at making clear roles of *loop groups* relating them [4]. Such an aspect is expected to give a new algebraic tool for microscopic understanding of the fermion systems.

To obtain a microscopic understanding of cooperative phenomena, the concept of collective motion is introduced in relation to TD variation of a self-consistent mean field, while that of independent-particle motion is described in terms of particles referring to a stationary mean field. The TD variation of the TD mean field is attributed to couplings between the collective and the independent-particle motions and couplings among quantum fluctuations of the TD mean field [5]. In viewing the above picture for both the motions, we have a very important and interesting question, which motivates this paper: what algebraic mechanism could play behind the concepts mentioned above to obtain the unified aspect of both the motions? In this paper, we will investigate profoundly the TDHF equation on circle S^1 presented in I

$$\begin{aligned} &\mathrm{i}\hbar\partial_t U(g_u(t,z,z^{-1}))|\phi\rangle = H_{\mathrm{HF}}[W(g_{su}(t,z,z^{-1}))]U(g_u(t,z,z^{-1}))|\phi\rangle \\ &g_u(t,z,z^{-1}) = g_{su}(t,z,z^{-1})\mathrm{e}^{-\mathrm{i}\mathfrak{a}(t,z,z^{-1})/\hbar} \end{aligned} \tag{1.1}$$

where $g_u \in U(n)$ and $g_{su} \in SU(n)$. $|\phi\rangle$ is an m-particle Slater determinant (S-det) called a simple state. The $\mathfrak{a}$ denotes a real valued n-dimensional diagonal matrix. The W defined later denotes a density matrix dependent on t and S^1 ($z = \mathrm{e}^{\mathrm{i}\varphi}$) through g_{su}. An answer to the above question may be possible with the aid of algebraic reduction of u_n to su_n in the τ-functional method [6].

Following the conventional manner we start with an su_n Lie algebra consisting of a particle–hole (p–h) component and a u_n which includes particle–particle (p–p) and hole–hole (h–h) ones for the m simple state $|\phi\rangle$ [1]. Then we have a $U(n)$ group orbit. The equivalence relation identifies different states of phase depending only on diagonal components in h–h type with the same state [7]. Under this equivalence relation, it is essentially enough to treat only an $SU(n)$ group orbit. The HF Hamiltonian, however, has a value on the u_n but not on the su_n. Therefore, to describe dynamics on the $SU(n)$ group orbit, we must remove extra components not satisfying the su_n from the fully parametrized HF Hamiltonian. Using the equivalence relation we have only enough to take diagonal components in p–p and h–h types into account to assign the HF Hamiltonian. Each value of these quantities plays a role of the phase of the (quasi-particle) fermion gauge as the canonical transformation shows. For instance, let $c_\alpha^\dagger(g_{su})$ and $c_\alpha(g_{su})$ ($\alpha = 1, \ldots, n$) be (quasi-particle) fermion creation and annihilation operators. Then $c_\alpha^\dagger(g_{su}) \simeq c_\alpha^\dagger(g_{su})\mathrm{e}^{-\mathrm{i}(\mathfrak{a})_{\alpha\alpha}}$ and h.c. hold. It is noted that these quantities play crucial roles to elucidate a unified algebraic understanding of the concepts of particle and collective modes in studying relations between the SCF method and the τ-functional method shown in I. The gauge-phase can be separated into a term which comes from single pair of fermions and another term originated from a particle-number operator of the fermions on the Lie algebra of the fermion pairs. The former relates to the particle mode and the latter to the collective one. By removing the above superfluous components of the HF Hamiltonian, it turns out that assignment of the HF Hamiltonian to both the modes brings the concepts of particle and collective motions. The usual TDHF theory does not have a complete scheme to treat both the motions in a unified manner. On the contrary, the TDHF equation on S^1 (1.1) has a powerful scheme for such problems. It provides not only a manifest and algebraic understanding of the motions but also a theoretical scheme to describe large-amplitude collective motion with a single pair of collective variables. As a simple illustration of particle and collective motions, assuming time-periodicity of the motions, we can derive a new unified equation for both the motions which goes beyond the usual HF and RPA equations.

The TDHF theory on S^1 is constructed on a collection of infinitely various subgroup orbits consisting of *loop paths* in a finite-dimensional Grassmannian Gr_m of finite-dimensional fermion Fock space. As a consequence, it will be exhibited that the TDHF theory on S^1 can be also built on an infinite-dimensional Grassmannian of infinite-dimensional fermion Fock space F_∞ [3]. To do this the above particle-number operator must be proved to be the set of shift operators with degree $nr (r \in \mathbb{Z})$ in the τ-functional method [8]. Simultaneouly we must extract su_n-components from the fully parametrized HF Hamiltonian by the above assignment. This is nothing but the su_n $(\in sl_n)$ reduction in the τ-functional method. Suppose that the m simple state $|\phi\rangle$ corresponds to a highest weight vector according to the idea of Dirac's positron theory [9]. If we choose a vacuum with broken symmetry on S^1, the vacuum state is able to deform through the shift operators associated with collective modes, so the τ-function in the soliton theory can be also deformed. In other words, by varying weight functions attached to those operators, which originate from the above assignment, the vacuum state can really move to change drastically. Then we find the algebraic mechanism for appearance of the collective motion induced by TD mean-field potential. This also simply and clearly gives us a deeper algebraic understanding of physical concepts of symmetry breaking and successive occurrence of collective motion due to recovery of the symmetry. To approach a solution of the new TDHF equation, we must investigate seriously how to extract various subgroup orbits satisfied with the Plücker relation on S^1, i.e. submanifolds in the infinite-dimensional Grassmannian in F_∞. This is connected with a problem of how to determine a solution form for the soliton equation in the su_n $(\in sl_n)$-hierarchy.

In section 2, manipulating algebra with the aid of *loop groups*, a new theory for describing both the motions in a unified way is given. In section 3, we strictly treat it using affine Kac–Moody algebra and transcribe it onto the τ-functional space. Time evolution of the vacuum state (mean-field potential) is explicitly given for a simple stationary solution. An embedded form of the usual RPA equation and a corresponding collective Hamiltonian are shown. Finally, a summary and some concluding remarks will be given.

2. Algebraic mechanism causing particle and collective motions

Let us start with the TDHF theory having a group parameter on circle S^1 $(z = \mathrm{e}^{\mathrm{i}\varphi})$

$$\mathrm{i}\partial_t g_u(t, z, z^{-1}) = \mathcal{F}[W(g_{su}(t, z, z^{-1}))]g_u(t, z, z^{-1}) \tag{2.1}$$

where we have used $\hbar = 1$. Symbols $\dagger$ and $\star$ mean Hermitian and complex conjugations, respectively. The HF Hamiltonian $\mathcal{F}_{\alpha\beta}$ $(\alpha,\ \beta = 1, \ldots, n)$ is represented as

$$\mathcal{F}[W(g_{su})]_{\alpha\beta} = h_{\alpha\beta} + [\alpha\beta|\gamma\delta]W_{\delta\gamma} \qquad W_{\alpha\beta} \stackrel{d}{=} \sum_{a=1}^{m} (g_{su})_{\alpha a}(g_{su}^{\dagger})_{a\beta}. \tag{2.2}$$

The above TDHF equation on S^1 can be rewritten into

$$\begin{aligned}
&\mathrm{i}\partial_t g_{su}(t, z, z^{-1}) = \mathcal{F}_{su}(t, z, z^{-1})g_{su}(t, z, z^{-1}) \\
&\mathcal{F}_{su}(t, z, z^{-1}) \stackrel{d}{=} \mathcal{F}[W(g_{su}(t, z, z^{-1}))] - g_{su}(t, z, z^{-1}) \cdot \partial_t \mathfrak{a}(t, z, z^{-1}) \cdot g_{su}^{\dagger}(t, z, z^{-1}).
\end{aligned} \tag{2.3}$$

Since the $\mathcal{F}_{su}$ must be elements of su_n, the relation below holds:

$$\mathrm{Tr}\left\{\mathcal{F}[W(g_{su}(t, z, z^{-1}))] - g_{su}(t, z, z^{-1}) \cdot \partial_t \mathfrak{a}(t, z, z^{-1}) \cdot g_{su}^{\dagger}(t, z, z^{-1})\right\} = 0 \tag{2.4}$$

which leads to $\mathrm{Tr}\,\mathcal{F}[W(g_{su}(t, z, z^{-1}))] = \mathrm{Tr}\,\partial_t \mathfrak{a}(t, z, z^{-1})$ where Tr denotes the operation of trace summation. We call this relation the su_n-condition for the HF Hamiltonian. In the usual treatment of the SCF method which is not explicitly dependent on z, the HF solution is taken to be $g_{su} = g^0_{su}$ ($\partial_t g^0_{su} = 0$) and $\mathbb{a}(t) = \mathbb{a}^0 \cdot t$. Then the su_n-condition becomes $\mathrm{Tr}\,\mathcal{F} = \mathrm{Tr}\,\mathbb{a}^0$. Each component of $\mathbb{a}^0$ is a well known quasi-particle energy. It brings time evolution of phase of the (quasi-particle) fermion gauge. For a general solution, time evolution of phase with respect to an infinitesimal interval $\mathrm{d}t$ can be expressed as $\partial_t \mathbb{a}(t) \cdot \mathrm{d}t$. Throughout the study of relations between the SCF method and the τ-functional method in I, we notice the following facts. The m simple state $|\phi\rangle$ corresponds to a highest weight vector in F_∞ and leads us to choice of a vacuum with broken symmetry. The shift operators with degree nr ($r \in \mathbb{Z}$) given later play a crucial role for the collective mode. They are related to the unit matrix I_n in the Lie algebra and the particle-number operator in the usual SCF method. These facts, as is shown in the succeeding section, can separate the above phase of fermion guage into a particle mode and a collective one.

From the three points (1) the su_n-condition for the HF Hamiltonian, (2) the vacuum state given by the idea of Dirac's positron theory and (3) the phase of (quasi-particle) fermion gauge $\mathbb{a}$ separated into particle and collective modes, we can elucidate a unified algebraic understanding of the concepts of particle and collective motions. We are now in a stage to describe manifestly the algebraic mechanism causing the concept of particle and collective motion. Using a formal Laurent expansion (Fourier), the group parameter on S^1 in (2.1) can be rewritten as

$$\begin{aligned} &g_{su}(t,z,z^{-1}) = \sum_{r\in\mathbb{Z}} g_{su}(t)_r z^r \qquad \mathbb{a}(t,z,z^{-1}) = \mathbb{a}_0(t) + \sum_{r\geqslant 1}\left(\varepsilon_r(t) I_n z^r + \varepsilon_{-r}(t) I_n z^{-r}\right) \\ &W(g_{su}(t,z,z^{-1})) = \sum_{r\in\mathbb{Z}} W_r(t) z^r \qquad (W_r)_{\alpha\beta} = \sum_{s\in\mathbb{Z}}\sum_{a=1}^{m}((g_{su})_s)_{\alpha a}((g^\dagger_{su})_{s-r})_{a\beta}. \end{aligned} \tag{2.5}$$

We call $\mathbb{a}_0$ a particle phase and $\varepsilon_r z^r$ with $\varepsilon_{-r}(t) = \varepsilon^\star_r(t)$ a collective phase, because the former is connected to single pair of fermions and the latter to a collective pair of fermions (the particle-number operator). Note that the latter itself does not mean the collective operator in the ordinary sense. Then with the use of $\hat{g}_{su}$ and $\hat{I}_n(r)$, which are defined by infinite periodic sequences of block form of g_{su} and n-dimensional unit matrix I_n [3,8], the TDHF equation (2.3) on S^1 can be transformed into

$$\begin{aligned} &D_t\hat{g}_{su} = \left\{\mathcal{F}[W(\hat{g}_{su})] - \hat{g}_{su}\cdot\partial_t\hat{\mathbb{a}}_0\cdot\hat{g}^\dagger_{su} + \mathrm{i}\sum_{r\geqslant 1}\left(D_{t,r}\varepsilon_r\cdot\hat{I}_n(r) + D_{t,-r}\varepsilon_{-r}\cdot\hat{I}_n(-r)\right)\right\}\hat{g}_{su} \\ &(\mathcal{F}_r)_{\alpha\beta} \overset{d}{=} h_{\alpha\beta}\cdot\delta_{r,0} + [\alpha\beta|\gamma\delta](W_r)_{\delta\gamma} \qquad D_{t,r} \overset{d}{=} \mathrm{i}\partial_t - r\partial_t\varphi(t) \end{aligned} \tag{2.6}$$

where we have used $[\hat{g}_{su}, \hat{I}_n(\pm r)] = 0$. The explicit block form is defined in I.

Defining $\hat{\varepsilon} \overset{d}{=} \sum_{r\geqslant 1}(\varepsilon_r\cdot\hat{I}_n(r) + \varepsilon_{-r}\cdot\hat{I}_n(-r))$, particle and collective phases are given as

$$\begin{aligned} &\hat{\mathbb{a}}_0 = \begin{bmatrix} \ddots & & & & \\ & \mathbb{a}_0 & & & \\ & & \mathbb{a}_0 & & \\ & & & \mathbb{a}_0 & \\ & & & & \ddots \end{bmatrix} \\ &\hat{\varepsilon} = \begin{bmatrix} & \ddots & & & & \ddots & \\ & \varepsilon_{-1}I_n & 0 & \varepsilon_1 I_n & & & \\ & & \varepsilon_{-1}I_n & 0 & \varepsilon_1 I_n & & \\ & & & \varepsilon_{-1}I_n & 0 & \varepsilon_1 I_n & \\ \ddots & & & & & \ddots & \end{bmatrix}. \end{aligned} \tag{2.7}$$

The time-dependence $\varphi = \varphi(t)$ on circle S^1 is not an absolutely necessary condition but gives important roles for collective motions in the SCF method. Due to this we can interpret the *collective motion* as resonating/interfering phenomena between fermions which are dependent on a common gauge factor on S^1.

Using the above framework, the su_n-condition for the HF Hamiltonian of (2.4) is given as

$$\begin{aligned} \mathrm{Tr}\,\mathcal{F}_0 &= \sum_{\alpha=1}^{n}\{h_{\alpha\alpha} + [\alpha\alpha|\gamma\delta](W_0)_{\delta\gamma}\} = \sum_{\alpha=1}^{n} \partial_t(\mathfrak{a}_0)_{\alpha\alpha} = \mathrm{Tr}\,\partial_t\mathfrak{a}_0 \\ \mathrm{Tr}\,\mathcal{F}_{\pm r} &= \sum_{\alpha=1}^{n}[\alpha\alpha|\gamma\delta](W_{\pm r})_{\delta\gamma} = -\mathrm{i}\sum_{\alpha=1}^{n} D_{t,\pm r}\varepsilon_{\pm r}(I_n)_{\alpha\alpha} = -\mathrm{i}nD_{t,\pm r}\varepsilon_{\pm r}. \end{aligned} \tag{2.8}$$

Substituting these differential terms of the collective phase into the original TDHF equation (2.6) on S^1, it can be cast into

$$D_t\hat{g}_{su} = \left\{\mathcal{F}[W(\hat{g}_{su})] - \hat{g}_{su}\cdot\partial_t\hat{\mathfrak{a}}_0\cdot\hat{g}_{su}^{\dagger} - \frac{1}{n}\sum_{r\geqslant 1}\left(\mathrm{Tr}\,\mathcal{F}_r\cdot\hat{I}_n(r) + \mathrm{Tr}\,\mathcal{F}_{-r}\cdot\hat{I}_n(-r)\right)\right\}\hat{g}_{su} \tag{2.9}$$

where we have used $[\hat{g}_{su}, \hat{I}_n(\pm r)] = 0$ again. This TDHF equation on S^1 shows manifestly the dependence of both the phases on $\hat{g}_{su}$. While the motion of the first phase appears as that of the degrees of freedom of each (quasi-) particle, motion of the second appears as collective motion in common to all the particle ones. This equation has a scheme capable of describing large-amplitude collective motions if we could know a concrete form of $\hat{g}_{su}$.

We will now derive a new equation, which has a simple time-periodic solution. It is able to describe in a unified way both the motions going beyond the usual static HF equation and the RPA equation. If we take $\partial_t\hat{g}_{su}^0 = 0$, $\mathfrak{a}_0(t) = \mathfrak{a}_0^0\cdot t$ and $\varphi(t) = -\omega_c\cdot t$, the TDHF equation (2.9) on S^1 can be expressed as

$$\begin{aligned} &\hat{\mathfrak{a}}_0^0(\hat{g}_{su}^0) + \frac{1}{n}\sum_{r\geqslant 1}\left(\mathrm{Tr}\,\mathcal{F}_r\cdot\hat{I}_n(r) + \mathrm{Tr}\,\mathcal{F}_{-r}\cdot\hat{I}_n(-r)\right) + \omega_c\hat{g}_{su}^{0\,\dagger}\Gamma(\hat{g}_{su}^0) = \hat{g}_{su}^{0\,\dagger}\mathcal{F}(\hat{g}_{su}^0)\hat{g}_{su}^0 \\ &\Gamma(\hat{g}_{su}^0) \stackrel{d}{=} \begin{bmatrix} \ddots & & & & & \ddots \\ -g_{-1}^0 & 0 & g_1^0 & & & \\ & -g_{-1}^0 & 0 & g_1^0 & & \\ & & -g_{-1}^0 & 0 & g_1^0 & \\ \ddots & & & & & \ddots \end{bmatrix} \\ &(\mathrm{i}\partial_t \pm r\omega_c)\varepsilon_{\pm r} = \frac{\mathrm{i}}{n}\,\mathrm{Tr}\,\mathcal{F}_{\pm r}(\hat{g}_{su}^0). \end{aligned} \tag{2.10}$$

Notice that large-amplitude collective motion for only one mode should be described by only one angular frequency ω_c invariant under $\hat{g}_{su}^0$ but the quasi-particle energy is in general dependent on $\hat{g}_{su}^0$. From equation (2.10) we see the following: $\hat{\mathfrak{a}}_0^0$ relating to the particle phase gives a (quasi-) particle energy and $\mathrm{Tr}\,\mathcal{F}_{\pm r}$ ($r \geqslant 1$) give a collective energy due to time evolution of the collective phase. The latter are interpreted as an energy renormalized to the (quasi-) particle energy owing to the collective motion in the sense of the quantity $\oint\{\mathfrak{a}_0^0 + \frac{1}{n}\sum_{r\geqslant 1}\left(\mathrm{Tr}\,\mathcal{F}_r(\hat{g}_{su}^0)\mathrm{e}^{-\mathrm{i}r\omega_c t} + \mathrm{Tr}\,\mathcal{F}_{-r}(\hat{g}_{su}^0)\mathrm{e}^{\mathrm{i}r\omega_c t}\right) I_n\}\,\mathrm{d}t$ for a periodic interval. Each difference of energies in the transition, however, remains $(\mathfrak{a}_0^0)_{\alpha\alpha} - (\mathfrak{a}_0^0)_{\beta\beta}$ ($\alpha, \beta = 1, \ldots, n$) through alternative cancellation of the renormalized energies. Deformation of the vacuum state arising from the collective phase will be explicitly classified with the help of the Schur function [6, 8], but now it is possible that amplitudes of oscillation are obtained by solving differential equations for ε_r and ε_{-r} in (2.10) with appropriate initial conditions as shown later. The usual SCF method describing time-periodic collective motions, as the RPA equation does, is related to treatment of the term $\omega_c\hat{g}_{su}^{0\,\dagger}\Gamma(\hat{g}_{su}^0)$ in the LHS of the first equation in (2.10).

Turning back to the TDHF equation on S^1 (2.9), it is possible to elucidate an algebraic mechanism if we solve the following problems of symmetry breaking and successive occurrence of collective motion due to recovery of the symmetry: (i) Determination of a form of solution $\hat{g}_{su}$ linking to that of raising and decreasing states on circle S^1. (ii) Canonical transformation $U(\hat{\mathfrak{a}})$ by both the phases, which acts on the vacuum state and causes a broken symmetry on S^1. Then $U(\hat{\mathfrak{a}})$ can change not only phase in the ordinary sense but also magnitude of amplitude of the vacuum state. (iii) Appearance of motion of the collective phase satisfying the su_n-condition for the HF Hamiltonian accompanying change of the vacuum state. Thus it can be said that the collective motion is just a motion of the vacuum state. It should be noted that in the usual SCF manner a phase term only changes the so-called phase in a vacuum state. In contrast, in the SCF method on S^1 the same term produces not only change of the *phase* but also that of the *amplitude* of the vacuum state.

The TDHF theory on S^1 provides not only an algebraic means clarifying physical concepts for microscopic understanding of fermion many-body systems but also a scheme capable of describing large-amplitude collective motion for a single pair of collective variables. The theory is also able to give an interesting illustrative example to clear an algebraic structure among the original fermion field, the vacuum field defined in the SCF potential and the bosonic field associated with the Laurent spectra on S^1.

3. On affine Kac–Moody algebra $\widehat{su_n}$

We will give more strictly the algebraic mechanism causing both the motions with the help of an associative affine Kac–Moody algebra [8]. According to sections 1 and 2, with the use of a free-particle vacuum $|0\rangle$ ($c_\alpha|0\rangle = 0$), let the group orbit and m simple state be the following forms, respectively:

$$U(g_{su}\mathrm{e}^{-\mathrm{i}\mathfrak{a}})|\phi\rangle = U(g_{su})U(\mathrm{e}^{-\mathrm{i}\mathfrak{a}})|\phi\rangle \qquad |\phi\rangle = c_m^\dagger \cdots c_1^\dagger|0\rangle. \tag{3.1}$$

In the usual SCF method $U(\mathrm{e}^{-\mathrm{i}\mathfrak{a}})|\phi\rangle = \mathrm{e}^{-\mathrm{i}\sum_{a=1}^{m}(\mathfrak{a})_{aa}}|\phi\rangle \simeq |\phi\rangle$ denotes the equivalence relation mentioned before. The (quasi-) fermion operators become dependent on S^1 through g_{su}. Then $c^\dagger$ and c must be also dependent on S^1 from the viewpoint of symmetry, that is, they must also belong to elements in the (quasi-) particle. The Laurent expansion in section 2 induces these fermion operators so that infinite-dimensional fermion operators are introduced [3]. Let $\psi_{nr+\alpha}$ and $\psi^\star_{nr+\alpha}$ ($r \in \mathbb{Z}, \alpha = 1, \ldots, n$) be the infinite-dimensional fermion operators. On the circle S^1, z can be regarded as a common gauge factor of the infinite-dimensional fermion operators. It becomes a mathematical device to raise resonating (interfering) phenomena between fermions as a collective motion. To describe such phenomena, we must inevitably go beyond the conventional manner in the finite-dimensional fermion Fock space to the infinite one. For this aim, it is effective to introduce an affine Kac–Moody algebra according to the idea of Dirac's positron theory. Let $|\mathrm{Vac}\rangle$ and $|m\rangle$ be a perfect vacuum and a reference vacuum $|m\rangle = \psi_m \cdots \psi_1|\mathrm{Vac}\rangle$ [8]

$$\begin{aligned} &\psi_{nr+\alpha}|\mathrm{Vac}\rangle = 0 \qquad \langle\mathrm{Vac}|\psi^*_{nr+\alpha} = 0 \qquad (r \leqslant -1)\\ &\psi^*_{nr+\alpha}|\mathrm{Vac}\rangle = 0 \qquad \langle\mathrm{Vac}|\psi_{nr+\alpha} = 0 \qquad (r \geqslant 0). \end{aligned} \tag{3.2}$$

We embed the free vacuum $|0\rangle$ and the m simple state $|\phi\rangle$ into the infinite-dimensional Fock space F_∞ as $|0\rangle \mapsto |\mathrm{Vac}\rangle$ and $|\phi\rangle \mapsto |m\rangle (m = 1, \ldots, n)$. We are forced to choose a vacuum with broken symmetry on S^1. Using the correspondence $c^\dagger_\alpha c_\beta z^r \mapsto \tau(e_{\alpha\beta}(r)) \stackrel{d}{=} \sum_{s\in\mathbb{Z}} \psi_{n(s-r)+\alpha}\psi^*_{ns+\beta}$ and the normal-ordered product $\psi_{nr+\alpha}\psi^\star_{ns+\beta} :\stackrel{d}{=} \psi_{nr+\alpha}\psi^\star_{ns+\beta} - \delta_{\alpha\beta}\delta_{rs}$

($s < 0$), let us define an $\widehat{su_n}$ ($\subset \widehat{sl_n}$) Lie algebra $X_\gamma = \overline{X}_\gamma + \mathbb{C} \cdot c$ with $\mathbb{C}^* = -\mathbb{C}$, $c|m\rangle = 1 \cdot |m\rangle$ (level-one case) and the two-cocycle α $(= \sum_{-N}^{N} r \operatorname{Tr} \gamma_r \gamma'_{-r})$ [8] where

$$\begin{aligned}
&\overline{X}_\gamma = \sum_{r=-N}^{N} \sum_{s\in\mathbb{Z}} (\gamma_r)_{\alpha\beta} : \psi_{n(s-r)+\alpha} \psi^*_{ns+\beta} : \qquad \gamma_r^\dagger = -\gamma_{-r} \ (\operatorname{Tr} \gamma_r = 0) \\
&[X_\gamma,\ c] = 0 \qquad [X_\gamma,\ X_{\gamma'}] = \overline{X}_{[\gamma,\ \gamma']} + \alpha(\gamma, \gamma') \cdot c.
\end{aligned} \tag{3.3}$$

Under the equivalence $I_n z^{\pm r} \simeq \hat{I}_n(\pm r) \mapsto \Lambda_{\pm nr}$, the phase $\mathfrak{a}(t, z, z^{-1})$ in g_u of (2.1) corresponds as

$$\begin{aligned}
&\mathfrak{a}(t, z, z^{-1}) \mapsto X_{\mathfrak{a}} \overset{d}{=} X_{\mathfrak{a}_0} + X_\varepsilon \\
&X_{\mathfrak{a}_0} \overset{d}{=} \sum_{\alpha=1}^{n} \sum_{s\in\mathbb{Z}} (\mathfrak{a}_0(t))_{\alpha\alpha} : \psi_{ns+\alpha} \psi^*_{ns+\alpha} : \qquad X_\varepsilon \overset{d}{=} \sum_{r\geqslant 1} (\varepsilon_r(t) \Lambda_{nr} + \varepsilon_r^\star(t) \Lambda_{-nr}).
\end{aligned} \tag{3.4}$$

$\Lambda_{\pm nr}$, composed of the shift operators $\Lambda_k \overset{d}{=} \sum_{i\in\mathbb{Z}} : \psi_i \psi^*_{i+k}:$ giving the boson algebra, has properties

$$\begin{aligned}
&[\Lambda_{nr}, \Lambda_{ns}] = nr\delta_{r+s,0} \qquad [X_\gamma, \Lambda_{\pm nr}] = [X_{\mathfrak{a}_0}, \Lambda_{\pm nr}] = 0 \\
&\Lambda_{nr}|m\rangle = 0 \qquad (r \geqslant 1).
\end{aligned} \tag{3.5}$$

From these we obtain the action of the canonical transformation on the vacuum state $|m\rangle$ as

$$\begin{aligned}
&\mathrm{e}^{X_\gamma} \mathrm{e}^{-\mathrm{i}X_{\mathfrak{a}}} |m\rangle = \mathrm{e}^{X_\gamma} \mathrm{e}^{-\mathrm{i}X_{\mathfrak{a}_0}} \mathrm{e}^{-\mathrm{i}X_\varepsilon} |m\rangle = \mathrm{e}^{-\frac{1}{2}\sum_{r\geqslant 1} nr|\varepsilon_r|^2 - \mathrm{i}\sum_{a=1}^{m} (\mathfrak{a}_0)_{aa}} \mathrm{e}^{-\mathrm{i}\sum_{r\geqslant 1} \varepsilon_r^\star \Lambda_{-nr}} \mathrm{e}^{X_\gamma} |m\rangle \\
&\mathrm{e}^{-\mathrm{i}\sum_{r\geqslant 1} \varepsilon_r^\star \Lambda_{-nr}} = \sum_{r\geqslant 0} S_r(-\mathrm{i}\varepsilon_1^\star \Lambda_{-n}, -\mathrm{i}\varepsilon_2^\star \Lambda_{-2n}, \ldots)
\end{aligned} \tag{3.6}$$

where we have used the Baker–Campbell–Hausdorff formula $\mathrm{e}^A \mathrm{e}^B = \mathrm{e}^{A+B} \mathrm{e}^{\frac{1}{2}[A,B]}$ under $[A, B] \in c$-number and the Schur-polynomial, i.e. $\exp\left(\sum_{k\geqslant 1} x_k p^k\right) = \sum_{k\geqslant 0} S_k(x_1, x_2 \ldots) p^k$. Notice the equivalence relation $\Lambda_{-nr} \simeq \Lambda_{-nr} z^{-r}$. Then a canonical transformation by the particle phase induces change of only the phase of the vacuum state, but one by the collective phase deforms the vacuum state itself. The excited states on S^1 are also possibly classified in a subspace with degree nr by the Schur function; through that classification we become able to observe collective excitations. It should be noted that the collective term is independent of $\mathrm{e}^{X_\gamma} \in \widehat{SU}(n)$. We cannot distinguish raised states by Λ_{-nr} from each other. This means that those states are raised from either the original vacuum state $|m\rangle$ or from the (quasi-) particle state $\mathrm{e}^{X_\gamma}|m\rangle$ since the invariance relation $\mathrm{e}^{X_\gamma} \Lambda_{-nr} \mathrm{e}^{-X_\gamma} = \Lambda_{-nr}$ holds. Owing to this invariance property, *the collective motion arises through the su_n-condition of the HF Hamiltonian* (2.4). Then one can obtain the concepts of particle and collective motions. This statement just explains an algebraic mechanism working behind superficial observation for a unified understanding of both the particle and collective motions.

Next we observe the TDHF equation on τ-functional space. The τ-function should be a projection of the $\widehat{SU}(n)$ group orbit $\mathrm{e}^{X_\gamma}|m\rangle$ onto another group orbit of $\langle m|\mathrm{e}^{H(x)}$ ($H(x) = \sum_{j\geqslant 1} x_j \Lambda_j$ and $x = (x_1, x_2, \cdots)$), that is to say, $\tau_m(x, \hat{g}_{su}) \equiv \langle m|\mathrm{e}^{H(x)} \mathrm{e}^{X_\gamma}|m\rangle$. From $[X_\gamma, \Lambda_{nr}] = 0$ the τ-function is independent on x_{nr}. While the τ-functional method describes an x-flow by $H(x)$, the TDHF theory describes a t-flow of $\gamma(t)$ owing to the HF Hamiltonian satisfying the su_n-condition. This point is an essential difference between the two methods. Determination of solutions in the τ-functional method is nothing but that of subgroup orbits of the X_γ. In this sense, it is said that both the methods have treated common problems of how we could find forms of subgroup orbits truncated from a fully parametrized group manifold based on the Plücker relation. The above different point raises an interesting mathematical problem of the mean-field theory. Its construction on groups is made on maps from a group orbit onto another orbit of the same group under invariant group integration. Therefore the mean-field theory involves the generator coordinate (GC) method [10,11]. It may provide a nonlinear soliton-superposition method on a nonlinear space (Grassmannian) [10,12]. Standing on the viewpoint of local symmetry of infinite-dimensional fermion systems behind global symmetry of finite ones, we might rebuild the GC and nonlinear superposition methods by the infinite-dimensional shift operators. Then we will be able to study *concrete connections* between the methods in the mean-field theory and in the soliton theory [6,13,14].

Introducing a matrix $\mathcal{F}^c_{su}$, the TDHF equation (2.9) on S^1 can be simply rewritten as

$$\begin{aligned}
&\mathrm{i}\partial_t \hat{g}_{su} = \mathcal{F}_{su}(\hat{g}_{su})\hat{g}_{su} \\
&\mathcal{F}_{su}(\hat{g}_{su}) \stackrel{d}{=} \mathcal{F}[W(\hat{g}_{su})] - \mathcal{F}^c_{su}(\hat{g}_{su}) - \hat{g}_{su}\cdot\partial_t\hat{\mathfrak{a}}_0\cdot\hat{g}^\dagger_{su} \\
&\qquad\qquad -\frac{1}{n}\sum_{r\geqslant 1}\mathrm{Tr}\left(\mathcal{F}_r\cdot\hat{I}_n(r)+\mathcal{F}_{-r}\cdot\hat{I}_n(-r)\right) \\
&\{(\mathcal{F}^c_{su})_r\}_{\alpha\beta}(\hat{g},\partial_t\varphi) \stackrel{d}{=} -\partial_t\varphi\cdot\sum_{s\in\mathbb{Z}} s(g_s g^\dagger_{s-r})_{\alpha\beta}
\end{aligned}\tag{3.7}$$

which can be transcribed into the equation on τ-functional space [3] as

$$\begin{aligned}
&\mathrm{i}\partial_t\tau_m(x,\hat{g}_{su}(t)) = H_{F_\infty su}(x,\tilde{\partial}_x,\hat{g}_{su}(t))\tau_m(x,\hat{g}_{su}(t)) \\
&H_{F_\infty su}(x,\tilde{\partial}_x,\hat{g}_{su}(t)) = \sum_{r,s\in\mathbb{Z}}\{(\mathcal{F}_{su})_r(\hat{g}_{su}(t))\}_{\alpha\beta}\tilde{z}_{n(s-r)+\alpha,ns+\beta}(x,\tilde{\partial}_x)
\end{aligned}\tag{3.8}$$

where

$$\begin{aligned}
&\tilde{z}_{ij} \stackrel{d}{=} z_{ij} - \delta_{ij}\cdot 1 \qquad (j = \leqslant 0) \\
&z_{ij}(x,\tilde{\partial}_x) \stackrel{d}{=} \sum_{\mu,\nu\geqslant 0,k\geqslant 0} S_{i+k+\mu-m}(x)S_{-j-k+\nu+m}(-x)S_\mu(-\tilde{\partial}_x)S_\nu(\tilde{\partial}_x) \\
&\tilde{\partial}_x \stackrel{d}{=} \left(\frac{\partial}{\partial x_1},\frac{1}{2}\frac{\partial}{\partial x_2},\ldots\right).
\end{aligned}\tag{3.9}$$

Equation (3.8) is not dependent on a set of $\{x_{nr}\}$. We, however, become aware of the t-flow on the x_{nr} if we carefully pay attention to the su_n-condition for $\mathcal{F}_{su}(\hat{g}_{su})$ given through the second equation of (3.7). By using the boson mapping operator $\langle m|\mathrm{e}^{H(x)}$ with $H(x) = \sum_{k\geqslant 1;\neq nr} x_k\Lambda_k + \sum_{r\geqslant 1} x_{nr}\Lambda_{nr}$ (*Hamiltonian in τ-functional method*) and equations (3.5) and (3.6), a $\hat{u}_n$ group orbit of $\mathrm{e}^{X_\gamma}\mathrm{e}^{-\mathrm{i}X_{\mathfrak{a}}}|m\rangle$ can be mapped onto the τ-functional space as

$$\tau_m(x,\hat{g}_u) = \mathrm{e}^{-\frac{1}{2}\sum_{r\geqslant 1} nr|\varepsilon_r|^2 - \mathrm{i}\sum_{r\geqslant 1}\varepsilon^\star_r nr x_{nr}}\mathrm{e}^{-\mathrm{i}\sum_{a=1}^m(\mathfrak{a}_0)_{aa}}\tau_m(x,\hat{g}_{su}).\tag{3.10}$$

The second term of the first exponent in the RHS shows us the t-flow at each point on x_{nr} due to the TD nature of $\varepsilon^\star_r(\hat{g}_{su}(t))$. Then it is concluded as follows: while a particle motion appears as the t-flow of $\tau_m(x,\hat{g}_{su}(t))$ and also as the phase $\exp[-\mathrm{i}\sum_{a=1}^m\{\mathfrak{a}_0(\hat{g}_{su}(t))\}_{aa}]$, a collective motion appears as a motion of vacuum in the τ-function, which makes a change of amplitudes of both the vacuum and excited states. According to the usual SCF method if we attend only to the vacuum state, we see the collective motion is variation of the amplitude, which just corresponds to the motion of the TD mean-field potential.

Owing to the simplicity of the stationary solution of equation (2.10), we can prove the manifest existence of deformation and oscillation of the vacuum state and show how the usual RPA theory is embedded into the present new theory. From (2.10), solutions for $\varepsilon_r(t)$ and

$\varepsilon_r^\star(t)$ with constant terms $c_r(\hat{g}^0_{su})$ and $c_r^\star(\hat{g}^0_{su})$ are obtained as

$$\varepsilon_r(t) = \frac{\mathrm{i}}{nr\omega_c}\,\mathrm{Tr}\,\mathcal{F}_r(\hat{g}^0_{su}) + c_r(\hat{g}^0_{su})\mathrm{e}^{\mathrm{i}r\omega_c t} \qquad \varepsilon_r^\star(t) = -\frac{\mathrm{i}}{nr\omega_c}\,\mathrm{Tr}\,\mathcal{F}_{-r}(\hat{g}^0_{su}) + c_r^\star(\hat{g}^0_{su})\mathrm{e}^{-\mathrm{i}r\omega_c t} \tag{3.11}$$

$$|\varepsilon_r|^2 = \frac{1}{(nr\omega_c)^2}|\,\mathrm{Tr}\,\mathcal{F}_r(\hat{g}^0_{su})|^2 + |c_r(\hat{g}^0_{su})|^2 + \frac{\mathrm{i}}{n}\left\{\mathrm{Tr}\,\mathcal{F}_r(\hat{g}^0_{su})\cdot c_r^\star(\hat{g}^0_{su})\mathrm{e}^{-\mathrm{i}r\omega_c t} - \mathrm{c.c.}\right\} \tag{3.12}$$

where we have used $\mathrm{Tr}\,\mathcal{F}_{-r} = (\mathrm{Tr}\,\mathcal{F}_r)^\star$. Regarding the stationary group orbit $U(g_u)|\phi\rangle$ as $\mathrm{e}^{X_\gamma}\mathrm{e}^{-\mathrm{i}X_\epsilon}|m\rangle$ in (3.6) then we have

$$\begin{aligned}
\mathrm{e}^{X_\gamma}\mathrm{e}^{-\mathrm{i}X_\epsilon}|m\rangle = \exp&\left[-\sum_{r\geqslant 1}\left\{\frac{1}{nr\omega_c}(\mathrm{Tr}\,\mathcal{F}_r(\hat{g}^0_{su}))^\star + \mathrm{i}c_r^\star(\hat{g}^0_{su})\mathrm{e}^{-\mathrm{i}r\omega_c t}\right\}\Lambda_{-nr}\right] \\
&\times\exp\left[-\frac{1}{2}\sum_{r\geqslant 1}\left(\frac{1}{nr\omega_c^2}|\,\mathrm{Tr}\,\mathcal{F}_r(\hat{g}^0_{su})|^2 + nr|c_r(\hat{g}^0_{su})|^2\right.\right. \\
&\left.\left.+\mathrm{i}r\left\{\mathrm{Tr}\,\mathcal{F}_r(\hat{g}^0_{su})\cdot c_r^\star(\hat{g}^0_{su})\mathrm{e}^{-\mathrm{i}r\omega_c t} - \mathrm{c.c.}\right\}\right)\right] \\
&\times\mathrm{e}^{-\mathrm{i}\sum_{a=1}^m(\epsilon_0^0(\hat{g}^0_{su}))_{aa}t}\cdot U(\hat{g}^0_{su})|m\rangle.
\end{aligned} \tag{3.13}$$

In the above we have expressed the dependence of each quantity on $\hat{g}^0_{su}$ explicitly to show the relation between an $SU(n)$ group orbit and *particle and collective phases*. A function corresponding to the vacuum state, $\langle m|\mathrm{e}^{X_\gamma}\mathrm{e}^{-\mathrm{i}X_\epsilon}|m\rangle$, can be obtained, if we equate the RHS of the first line in the above to one using $\langle m|\Lambda_{-nr} = 0$. The usual SCF treatment in the finite many-body systems has never derived such a unified expression for particle and collective modes. It induces only the particle phase and not the collective one. Therefore it describes a collective motion as a recovery of a symmetry of the HF Hamiltonian since the vacuum state breaks the symmetry. Remember that we have selected a vacuum state $|m\rangle$ with broken symmetry on S^1. The shift operators Λ_{-nr} make an essential role for collective motions and behave as quantal collective boson operators rather than the ordinary RPA operator. The circle S^1 takes an active part in causing resonance (interference) between fermions. This has never been seen manifestly in the usual SCF method. As for the resonance, $\mathcal{F}^c_{su}$ in (3.7) produces elements of the vector field on $\hat{g}_{su}$ induced by a rotation of S^1 which corresponds to a motion of the common gauge phase φ of the infinite-dimensional fermion operators. Thus $\mathcal{F}^c_{su}$ plays a qualitative role of the collective HF Hamiltonian describing the resonating phenomena among the fermions. These phenomena take behaviours of a rigid motion occurring from only the term $\partial_t\varphi$ (remember the cranking model by Inglis [15]). The quantitative magnitude of the resonance is evaluated by distribution of values of the matrix $\mathcal{F}^c_{su}$. As a result the so-called collectivity in the usual SCF method is attributed to a geometrical property of Gr_m on S^1 which is independent from $\mathcal{F}[W(\hat{g}_{su})]$. This fact asserts that clarification of the algebro-geometrical structure of Gr_m on S^1 becomes a very important problem to construct various coordinate systems describing collective motions. $\partial_t\varphi$, i.e. ω_c means the component due to projection of $\mathcal{F}[W(\hat{g}_{su})]$ onto the $\mathcal{F}^c_{su}$. The above are the algebro-geometrical mechanism describing the symmetry breaking of fermion systems and successive occurrence of the collective motion due to the recovery of the symmetry.

To obtain solutions of the new equation (3.7) we must determine not only forms of subgroup orbits or the corresponding sub-Lie algebras but also initial conditions for $\varepsilon_r(t)$ and $\varepsilon_r^\star(t)$, whose conditions in the stationary solution correspond to the decision of $c_r(\hat{g}^0_{su})$ and $c_r^\star(\hat{g}^0_{su})$. $\hat{g}^0_{su}$ is a function on collective variables [3]. We should establish mathematical tools to extract them out of a fully parametrized TDHF manifold with the aid of the Plücker relation on S^1, i.e. the *soliton equation* derived in I. Further we have to clarify the relation between the above initial conditions and collective variables. Then one may find interesting illustrative problems for exploring algebraic relations of the fermion, boson and vacuum in the description for particle and collective motions and mean-field potential in condensed matter physics and nuclear physics. The expression for algebraic relations of the fermion and boson on the τ-functional space can be achieved by changing Λ_{-nr} to nrx_{nr} since the isomorphism σ_m; $F^{(m)}$ (m-charged fermion space) $\mapsto$ $B^{(m)}$ (corresponding boson space) is given as $\Lambda_{nr} \mapsto \frac{\partial}{\partial x_{nr}}$, $\Lambda_{-nr} \mapsto nrx_{nr}$ [8].

We will now embed the usual RPA equation in (2.10) onto F_∞ as follows: along the same line as the conventional manner, let us denote by g^0 and $\tilde{g}$ ($\in SU(n)$) a stationary solution describing an equilibrium state and a small fluctuational one around it, respectively. We here omit subscript su unless we have a fear of misunderstanding. The fluctuational term is approximated as $\tilde{g} \approx I_n + \eta^\star\theta z - \eta\theta^\dagger z^{-1} + \mathrm{O}(|\eta|^2)$ under the condition of a very small norm $\| \theta \| = |\eta| \approx 0$. For simplicity, we assume $\theta^\dagger$ and θ to be composed of only p–h type components as

$$\theta^\dagger \stackrel{d}{=} \begin{pmatrix} 0 & \varphi \\ \psi & 0 \end{pmatrix} \qquad \theta \stackrel{d}{=} \begin{pmatrix} 0 & \psi^\dagger \\ \varphi^\dagger & 0 \end{pmatrix}. \tag{3.14}$$

Then the density matrix and the HF Hamiltonian are approximated as

$$\begin{aligned} &W = W_0 + W_1 z + W_{-1}z^{-1} + \mathrm{O}(z^{\pm 2}) \qquad \mathcal{F} = \mathcal{F}_0 + \mathcal{F}_1 z + \mathcal{F}_{-1}z^{-1} + \mathrm{O}(z^{\pm 2}) \\ &W_0 = W_0^0 + \triangle W_0 \qquad \mathcal{F}_0 = \mathcal{F}_0^0 + \triangle\mathcal{F}_0 \end{aligned} \tag{3.15}$$

where

$$\begin{aligned} &(W_0^0)_{\alpha\beta} = \sum_{a=1}^{m} g^0_{\alpha a} g^{0\dagger}_{a\beta} \qquad \triangle W_0 = |\eta|^2 \sum_{i,j=m+1}^{n} g^0_{\alpha i}(\varphi^\dagger\varphi + \psi\psi^\dagger)_{ij} g^{0\dagger}_{j\beta} \\ &W_1 = \eta^\star g^0 \begin{pmatrix} 0 & -\psi^\dagger \\ \varphi^\dagger & 0 \end{pmatrix} g^{0\dagger} \qquad W_{-1} = \eta g^0 \begin{pmatrix} 0 & \varphi \\ -\psi & 0 \end{pmatrix} g^{0\dagger} \end{aligned} \tag{3.16}$$

$$\begin{aligned} &(\mathcal{F}_0^0)_{\alpha\beta} = h_{\alpha\beta} + [\alpha\beta|\gamma\delta](W_0^0)_{\delta\gamma} \qquad (\triangle\mathcal{F}_0)_{\alpha\beta} = [\alpha\beta|\gamma\delta](\triangle W_0)_{\delta\gamma} \\ &(\mathcal{F}_{\pm 1})_{\alpha\beta} = [\alpha\beta|\gamma\delta](W_{\pm 1})_{\delta\gamma}. \end{aligned} \tag{3.17}$$

The $\mathcal{F}_0^0$ is diagonalized as $g_0^{0\dagger}\mathcal{F}_0^0 g_0^0 = \epsilon_0^0$. We further define matrices of su_n-elements as

$$\tilde{\mathcal{F}}_{\pm 1} \stackrel{d}{=} \mathcal{F}_{\pm 1} - \frac{1}{n}\,\mathrm{Tr}\,\mathcal{F}_{\pm 1} \cdot I_n. \tag{3.18}$$

The trace terms give no influence on the elements of p–h type but due to them the diagonal elements of h–h type can be closed within the first order of η and $\eta^\star$, in a part of the RPA equation given below. Substituting these quantities into equation (2.10), we can easily find the RPA equation

$$\omega_c \begin{pmatrix} \ddots & & & & & \\ \eta\theta^\dagger & 0 & \eta^\star\theta & & & \\ & \eta\theta^\dagger & 0 & \eta^\star\theta & & \\ & & \eta\theta^\dagger & 0 & \eta^\star\theta & \\ & & & & \ddots & \end{pmatrix}$$

$$= \left[\begin{pmatrix} \ddots & & & \\ & \epsilon_0^0 & & & \\ & & \epsilon_0^0 & & \\ & & & \epsilon_0^0 & \\ & & & & \ddots \end{pmatrix}, \begin{pmatrix} \ddots & & & & & \\ -\eta\theta^\dagger & I_n & \eta^\star\theta & & & \\ & -\eta\theta^\dagger & I_n & \eta^\star\theta & & \\ & & -\eta\theta^\dagger & I_n & \eta^\star\theta & \\ & & & & & \ddots \end{pmatrix} \right]$$

$$
+\begin{pmatrix} \ddots & & & & \\ & g^{0\dagger} & & & \\ & & g^{0\dagger} & & \\ & & & g^{0\dagger} & \\ & & & & \ddots \end{pmatrix}
\begin{pmatrix} \ddots & & & & & \\ \tilde{\mathcal{F}}_{-1} & 0 & \tilde{\mathcal{F}}_{1} & & & \\ & \tilde{\mathcal{F}}_{-1} & 0 & \tilde{\mathcal{F}}_{1} & & \\ & & \tilde{\mathcal{F}}_{-1} & 0 & \tilde{\mathcal{F}}_{1} & \\ & & & & & \ddots \end{pmatrix}
$$

$$
\times \begin{pmatrix} \ddots & & & & \\ & g^{0} & & & \\ & & g^{0} & & \\ & & & g^{0} & \\ & & & & \ddots \end{pmatrix}
$$

$$
+ \text{higher order terms of } \mathrm{O}(\eta^2, \eta^{\star 2}, \eta\eta^\star, \eta^\star\eta). \tag{3.19}
$$

The collective HF Hamiltonian $\mathcal{F}^c$ and classical collective Hamiltonian $\mathcal{H}^c$ up to $|\eta|^2$ can be given as

$$
\mathcal{F}^c = \omega_c \hat{g}^0 \begin{pmatrix} \ddots & & & & & \\ \eta\theta^\dagger & |\eta|^2[\theta, \theta^\dagger] & \eta^\star\theta & & & \\ & \eta\theta^\dagger & |\eta|^2[\theta, \theta^\dagger] & \eta^\star\theta & & \\ & & \eta\theta^\dagger & |\eta|^2[\theta, \theta^\dagger] & \eta^\star\theta & \\ & & & & & \ddots \end{pmatrix} \hat{g}^{0\dagger} \tag{3.20}
$$

$$
\begin{aligned}
\mathcal{H}^c &= \langle m | U^{-1}(\hat{g}^0) \sum_{\alpha,\beta=1}^{n} \sum_{r=-1}^{1} \sum_{s\in\mathbb{Z}} (\mathcal{F}^c_r)_{\alpha\beta} : \psi_{n(s-r)+\alpha} \psi^\star_{ns+\beta} : U(\hat{g}^0) | m \rangle \\
&= \omega_c \eta^\star \eta \langle m | \sum_{\alpha,\beta=1}^{n} \sum_{s\in\mathbb{Z}} [\theta, \theta^\dagger]_{\alpha\beta} : \psi_{ns+\alpha} \psi^\star_{ns+\beta} : | m \rangle = \omega_c \eta^\star \eta
\end{aligned}
$$

where we have used (3.3) and imposed a weak orthogonality condition as we did in the usual SCF treatment. Through the procedure developed up to now, we can see an important role of circle S^1 for appearance of collective motions rather than that by collective variables $\eta^\star$ and η. Then the Fock space of finite-dimensional fermion systems has the algebraic structure to be embedded into that of *infinite-dimensional fermion ones*.

A spectral parameter of the iso-spectral equation in the soliton theory and collective variables in the SCF method, though showing different aspects at a glance, work as scaling parameters on S^1. The former relates to scaling in description by analytical continuation of S^1, i.e. z. The latter play the role of deformation parameters of *loop paths* in the finite-dimensional Grassmannian Gr_m. They are also regarded as group parameters specifying group symmetry of the SCF equation [3, 16] and become a coordinate system on the collective submanifold [2]. We will discuss elsewhere clearly relations between the spectral parameter and the collective variables, which concern decision of initial conditions for $\varepsilon_r(t)$ and $\varepsilon_r^\star(t)$. We will also establish the algorithm to obtain subgroup manifolds or subalgebras based on the Plücker relation on S^1. In concrete applications to physics, we must take consideration of multifariously parametrized collective motions beyond description by a single pair of collective variables. The TDHF theory on S^1 is extended inevitably to that on multi-circles. Then we meet with a problem of how the Plücker relation on the multi-circles is constructed, which connects to a multi-component soliton theory [17].

Suppose the existence of a vacuum with broken symmetry hidden behind the symmetry. How does one take analytical tools for problems of broken symmetry in nonlinear physics into the SCF method? Very recently another infinite-dimensional algebraic approach has been proposed by using the Bethe ansatz (BA) wavefunction [18] and exact solutions for the $SU(2)$ Lipkin–Meshkov–Glick (LMG)-model Hamiltonian [19], the $SU(2)$ pairing one etc are described in a language of the infinite-dimensional Lie algebra [20]. This algebra is constructed by a power series expansion of the finite-dimensional Lie algebra with respect to parameters involved in the Hamiltonians. This becomes an infinitesimal form of the corresponding nonlinear algebra generated by building blocks of the BA wavefunctions [21]. On the other hand, we have introduced the infinite-dimensional Lie algebra from the viewpoint of *loop group*. How do we relate the two methods to each other? A further study should be pursued to obtain a collective submanifold decided by the SCF Hamiltonian with the use of the $SU(2)$ LMG-model Hamiltonian. Then we will acquire a clue to build a relation between the method using the Gaudin model [21, 22] and the present method.

4. Summary and concluding remarks

In I, to overcome the perturbative method, we have aimed at rebuilding the TDHF theory along the soliton theory in F_∞ and have shown manifestly that the framework of the theory turns out to be a new tool for microscopic understanding of fermion many-body systems. It elucidates simply and clearly the algebraic mechanism working behind the usual concepts of particle and collective motions of the systems, standing on the infinite-dimensional fermions. It also provides the physical meaning of a circle S^1, though being artificial in the soliton theory. The present circle is interpreted as a common gauge factor of fermions by which we can see resonating/interfering phenomena as collective motions.

The intimate relation of the SCF theory to the soliton theory comes from ways of constructing a closed system of soliton solution spaces. Hitherto the usual SCF method has been almost devoted to approach to such resonating phenomena in finite fermion systems. We must contrive construction of optimal coordinate systems on a group manifold on the basis of a useful principle. For this purpose the relation between the boson expansion method for the finite fermion systems and the τ-functional method for the infinite ones should be intensively investigated to clarify algebro-geometric structures of integrable systems. Such algebro-geometric approaches will achive connection between finite- and infinite-dimensional fermion systems. Various physical concepts and mathematical methods in the usual SCF theory may work well also in the infinite-dimensional ones. The SCF method mainly based on global symmetry so far should be much improved, noticing the local symmetry of the infinite-dimensional ones. Then the SCF method on S^1 may be expected to open a new area in vigorous pursuit of wider fields of physics. The GC method may provide soliton-superposition principles on nonlinear space (Grassmannian) [10, 12]. From the viewpoint of local symmetry of infinite-dimensional fermion systems behind global symmetry of finite ones, it is possible to reconstruct the GC and nonlinear superposition methods using the infinite-dimensional shift operators. Then we will be able to study the superposition method in soliton theory, which allows an exact nonlinear superposition principle [6, 13].

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